Solutions to Maths workbook - 2 | Permutation & Combination

Level – 3	Daily Tutorial Sheet – 15

234.(640)

Select any i^{th} vertex of the polygon and then select a j^{th} vertex and then a K^{th} vertex. Let there be x_1 vertices between the i^{th} and the j^{th}, x_2 between the j^{th} and the k^{th}, x_3 between k^{th} and i^{th}.
Clearly, $x_1 + x_2 + x_3 = 17$

Where, $x_i \geq 1$

Total number of integral solutions under such conditions $= {}^{17-1}C_{3-1} = {}^{16}C_2 = 120$.

Total number of triangles having no side common with the polygon $= \dfrac{{}^{20}C_1 \cdot 120}{3} = 800$.

Total number of isosceles triangles can be found out by fixing a vertex. For one vertex there are 9 isosceles triangles, but 1 triangle has 2 sides common with the polygon.
Total number of isosceles triangles $= 8 \times 20 = 160$.
Therefore, total number of such triangles = 640.

235. Consider the sum $1 + 1 + 1 + 1 + \ldots.. + 1$,
There being n terms in all. We can break this sum into one or more parts (n parts at the most) by either putting or not putting parenthesis after the $n - 1 '+'$ signs. This can be done in 2^{n-1} ways.

236.(133)

Let us play a seven round elimination tournament

First round: 64 objects eliminated

Second row: 32 eliminated and so on.

In seven rounds, 127 comparisons are made, and the heaviest object is identified along with the candidates for 2nd heaviest object: the seven objects that lost, one in each round, to the rank 1 object. These seven candidates play an elimination tournament to find the winner in six comparisons. Thus, total number of comparisons required $= 127 + 6 = 133$.

237. $\binom{n}{k}\binom{n+2}{k} \leq \binom{n+1}{k}^2$

Consider a plane with lattice points marked using a specific colour.

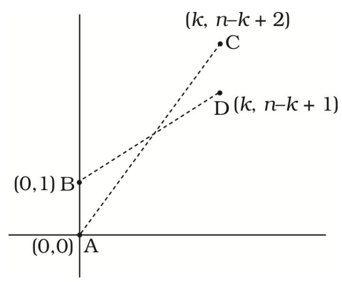
Note: A lattice point is the one having its co-ordinates as integral points.

Consider the point A (0, 0), B (0, 1), $C(k, n-k+2)$ and $D(k, n-k+1)$

The number of paths from A to D $= \binom{n+1}{k}$

The number of paths from B to C $= \binom{n+1}{k}$

The number of paths from A to C $= \binom{n+2}{k}$



The number of paths from B to D $=\binom{n}{k}$

Consider the set F of cartesian product of all the paths from A to D and B to C. Obviously the total number of elements in F $=\binom{n+1}{k}^2$

Consider the set G of cartesian product of all the paths from A to C and B to D. Obviously the total number of elements in G $=\binom{n+2}{k}\binom{n}{k}$.

Now, we can prove that there's an injection from g to f. (Hint: For every pair of paths P_1 from A to C and P_2 from B to D, P_2 must intersect P_1 at some lattice point).

Therefore, $\binom{n+2}{k}\binom{n}{k} \le \binom{n+1}{k}^2$.

238. For $k \in \{1, 2, 3, ..., n\}$, Let A_k be the set of all permutation of $\{1, 2, ...2n\}$ with k and $k+n$ in neighboring positions. For the set $A = \bigcup_{k=1}^{n} A_k$ of all pleasant permutations, the principle of inclusion exclusion yields $|A| = \sum_{n=1}^{n} |A_k| - \sum_{1 \le k < \ell \le n} |A_k \cap A_\ell| + ...$

This is a series of monotonically decreasing alternating terms. Hence, $|A| \ge \sum_{k=1}^{n} |A_k| - \sum_{1 \le k < \ell \le n} |A_k \cap A_\ell|$. We have $|A_k| = 2(2n-1)!$ because there are $(2n-1)!$ possibilities to arrange the elements $x \ne k$, $x \in \{1, 2, ..., 2n\}$ and two possibilities for the order $(k, k+n)$ *or* $(k+n, k)$. We have $|A_k \cap A_\ell| = 2^2\,(2n-2)!$. Indeed there are $(2n-2)!$ possibilities to arrange the $(2n-2)$ objects $x \ne k$, $x \ne \ell$ and then, 2^2 possibilities for the order of the two pair $\{k, k+n\}$ or $\{\ell, \ell+n\}$. Thus, we get $|A| \ge {}^nC_1 \times 2\left(2n-1\right)! - {}^nC_2 \times 2^2(2n-2)! > \frac{(2n)!}{2}$.

239. $2^{n-3}n\left(n+3\right)$

Let g_n be the number of ways one can plan such a semester. Let A(x), B(x), and C(x) be the generating functions for the sequences for the three individual tasks. That is $A(x) = \sum_{n \ge 0} 2^n x^n = \frac{1}{1-2x}$ since there are 2^n ways to choose an unspecified number of holidays from a set of n days. The number of subsets of [n] that are of odd size is 2^{n-1} if $n \ge 1$, and 0 if n = 0. Therefore, $B(x) = \sum_{n \ge 1} 2^{n-1} x^n = \frac{x}{1-2x}$. The number of subsets of [n] that are of even size is 2^{n-1} if $n \ge 1$, and 1 if n = 0. Therefore, $C(x) = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x}$. Now let $G(x)$ be the generating function of the sequence $\{g_n\}$. Then $G(x) = A(x)B(x)C(x)$.

Therefore, $G(x) = A(x)B(x)C(x) = \frac{1}{1-2x} \cdot \frac{x}{1-2x} \cdot \frac{1-x}{1-2x} = \frac{x(1-x)}{(1-2x)^3}$.

The partial fraction decomposition leads to the equation $G(x) = -\frac{1}{4} \cdot \frac{1}{1-2x} + \frac{1}{4} \cdot \frac{1}{(1-2x)^3}$.

Finally, using the binomial theorem, we get that $(1-2x)^{-3} = \sum_{n \ge 0} \binom{-3}{n} (-2x)^n = \sum_{n \ge 0} \binom{n+2}{2} 2^n x^n$

Therefore, $G(x) = -\frac{1}{4}\left(\sum_{n \ge 0} 2^n x^n\right) + \frac{1}{4}\left(\sum_{n \ge 0} \binom{n+2}{2} 2^n x^n\right)$.

So, $g_n = \left(\binom{n+2}{2} 2^n - 2^n / 4 = 2^{n-3} n(n+3), \text{ for } n \ge 0\right)$.